Mathematics Learning Centre

The University of Sydney

Functions: The domain and range

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1 Functions

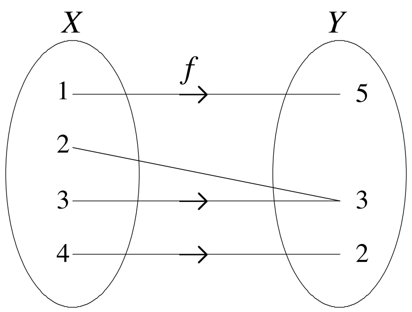
In these notes we will cover various aspects of functions. We will look at the definition of a function, the domain and range of a function, and what we mean by specifying the domain of a function.

1.1 What is a function?

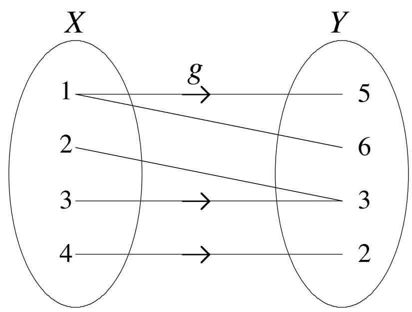
1.1.1 Definition of a function

> A function f from a set of elements X to a set of elements Y is a rule that assigns to each element x in X exactly one element y in Y.

One way to demonstrate the meaning of this definition is by using arrow diagrams.



$f : X \to Y$ is a function. Every element in X has associated with it exactly one element of Y.



$g : X \to Y$ is not a function. The element 1 in set X is assigned two elements, 5 and 6 in set Y.

A function can also be described as a set of ordered pairs (x, y) such that for any x-value in the set, there is only one y-value. This means that there cannot be any repeated x-values with different y-values.

The examples above can be described by the following sets of ordered pairs.

F = {(1,5),(3,3),(2,3),(4,2)} is a function.

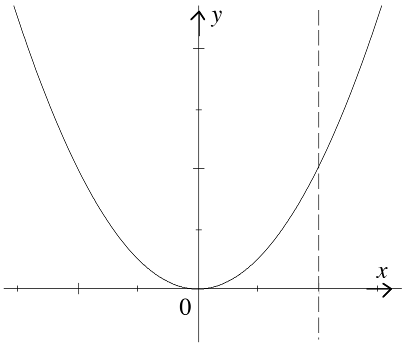
G = {(1,5),(4,2),(2,3),(3,3),(1,6)} is not a function.

The definition we have given is a general one. While in the examples we have used numbers as elements of X and Y, there is no reason why this must be so. However, in these notes we will only consider functions where X and Y are subsets of the real numbers.

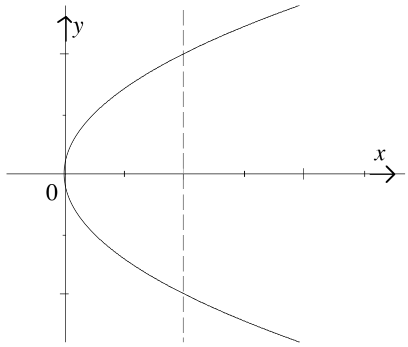
In this setting, we often describe a function using the rule, $y = f(x)$, and create a graph of that function by plotting the ordered pairs $(x, f(x))$ on the Cartesian Plane. This graphical representation allows us to use a test to decide whether or not we have the graph of a function: The Vertical Line Test.

1.1.2 The Vertical Line Test

The Vertical Line Test states that if it is *not possible* to draw a vertical line through a graph so that it cuts the graph in more than one point, then the graph *is* a function.



This is the graph of a function. All possible vertical lines will cut this graph only once.



This is not the graph of a function. The vertical line we have drawn cuts the graph twice.

1.1.3 Domain of a function

For a function $f : X \to Y$ the *domain* of f is the set X.

This also corresponds to the set of x-values when we describe a function as a set of ordered pairs (x, y).

If only the rule $y = f(x)$ is given, then the domain is taken to be the set of all real x for which the function is defined. For example, $y = \sqrt{x}$ has domain; all real $x \geq 0$. This is sometimes referred to as the *natural* domain of the function.

1.1.4 Range of a function

For a function $f : X \to Y$ the *range* of f is the set of y-values such that $y = f(x)$ for some x in X.

This corresponds to the set of y-values when we describe a function as a set of ordered pairs (x, y). The function $y = \sqrt{x}$ has range; all real $y \geq 0$.

Example

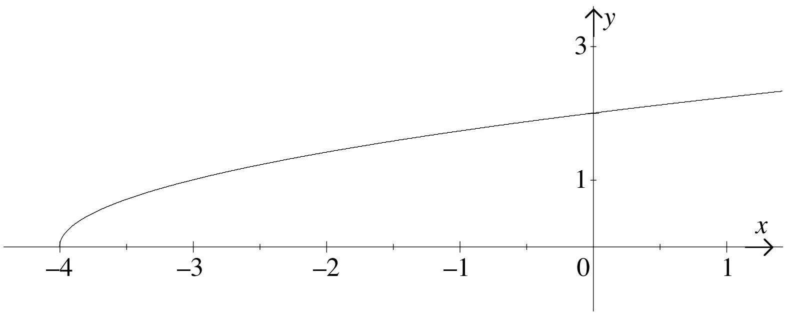
a. State the domain and range of $y = \sqrt{x+4}$.

b. Sketch, showing significant features, the graph of $y = \sqrt{x+4}$.

Solution

a. The domain of $y = \sqrt{x+4}$ is all real $x \geq -4$. We know that square root functions are only defined for positive numbers so we require that $x + 4 \geq 0$, ie $x \geq -4$. We also know that the square root functions are always positive so the range of $y = \sqrt{x+4}$ is all real $y \geq 0$.

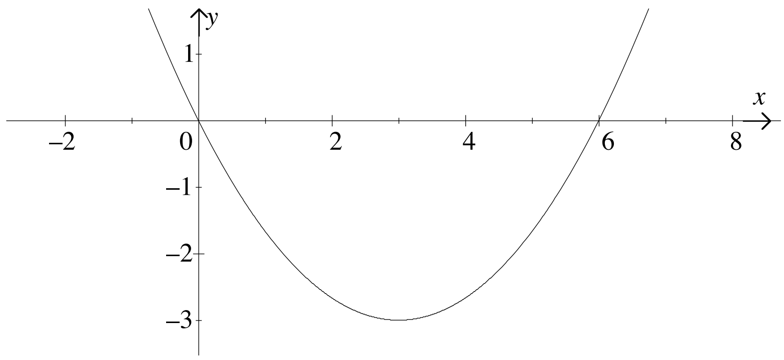
b.



The graph of $y = \sqrt{x+4}$.

Example

a. A parabola, which has vertex $(3, -3)$, is sketched below.



b. Find the domain and range of this function.

Solution

The domain of this parabola is all real x. The range is all real $y \geq -3$.

Example

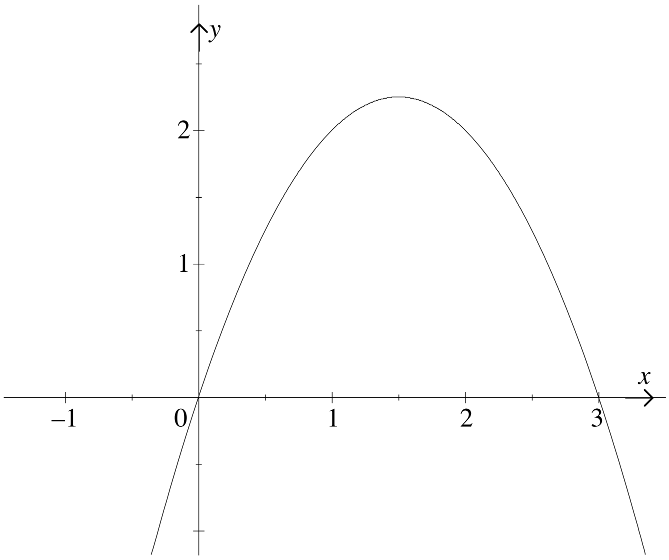
Sketch the graph of $f(x) = 3x - x^2$ and find

a. the domain and range

b. $f(q)$

c. $f(x^2)$.

Solution



The graph of $f(x) = 3x - x^2$.

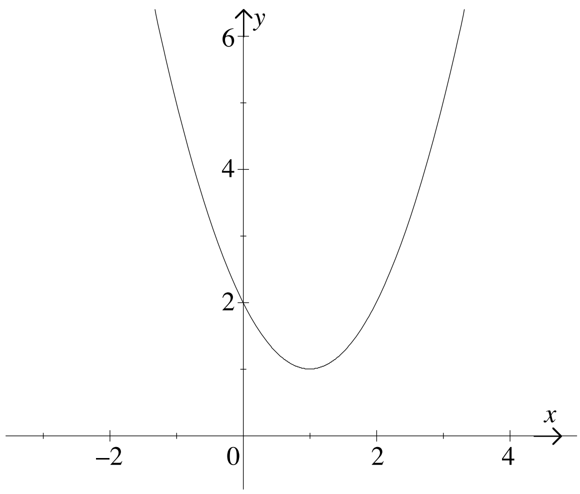
a. The domain is all real x. The range is all real y where $y \le 2.25$.

b. $f(q) = 3q - q^2$

c. $f(x^2) = 3(x^2) - (x^2)^2 = 3x^2 - x^4$

Example

The graph of the function $f(x) = (x-1)^2 + 1$ is sketched below.



The graph of $f(x) = (x-1)^2 + 1$.

State its domain and range.

Solution

The function is defined for all real x. The vertex of the function is at $(1,1)$ and therfore the range of the function is all real $y \geq 1$.

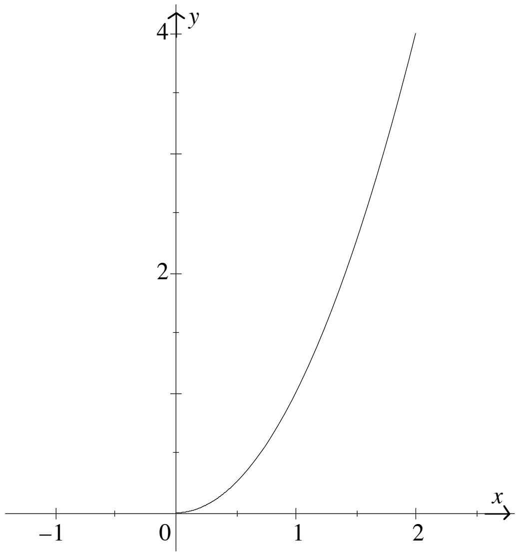
1.2 Specifying or restricting the domain of a function

We sometimes give the rule $y = f(x)$ along with the domain of definition. This domain may not necessarily be the natural domain. For example, if we have the function

$$y = x^2 \quad \text{for} \quad 0 \leq x \leq 2$$

then the domain is given as $0 \leq x \leq 2$. The natural domain has been restricted to the subinterval $0 \leq x \leq 2$.

Consequently, the range of this function is all real y where $0 \leq y \leq 4$. We can best illustrate this by sketching the graph.



The graph of $y = x^2$ for $0 \leq x \leq 2$.

1.3 Exercises

1. **a.** State the domain and range of $f(x) = \sqrt{9 - x^2}$.

b. Sketch the graph of $y = \sqrt{9 - x^2}$.

2. Sketch the following functions stating the domain and range of each:

a. $y = \sqrt{x - 1}$

b. $y = |2x|$

c. $y = \frac{1}{x-4}$

d. $y = |2x| - 1$.

3. Explain the meanings of function, domain and range. Discuss whether or not $y^2 = x^3$ is a function.

4. Sketch the following relations, showing all intercepts and features. State which ones are functions giving their domain and range.

a. $y = -\sqrt{4 - x^2}$

b. $|x| - |y| = 0$

c. $y = x^3$

d. $y = \frac{x}{|x|}, x \neq 0$

e. $|y| = x$.

5. Write down the values of x which are not in the domain of the following functions:

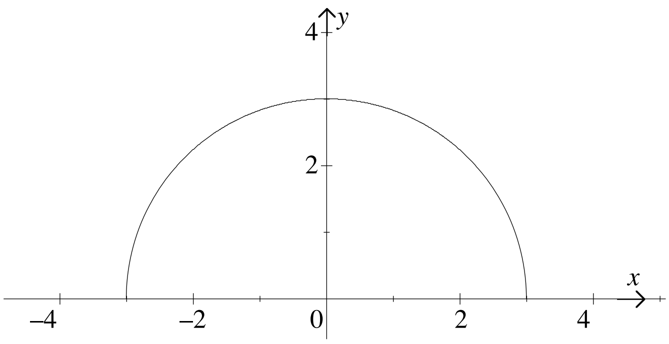
a. $f(x) = \sqrt{x^2 - 4x}$

b. $g(x) = \frac{x}{x^2-1}$

1.4 Solutions to exercises 1.3

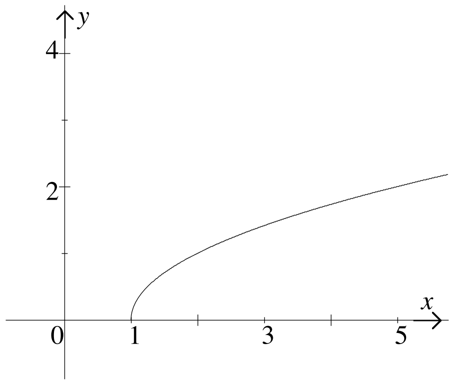
1. a. The domain of $f(x) = \sqrt{9 - x^2}$ is all real x where $-3 \le x \le 3$. The range is all real y such that $0 \le y \le 3$.

b.



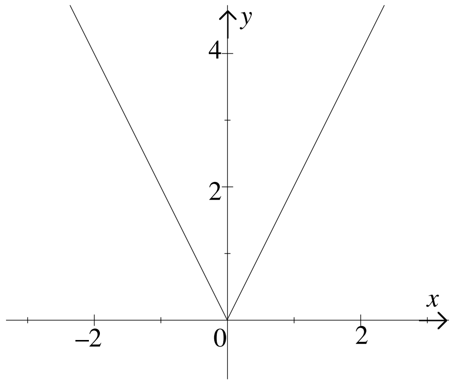
The graph of $f(x) = \sqrt{9 - x^2}$.

2. a.



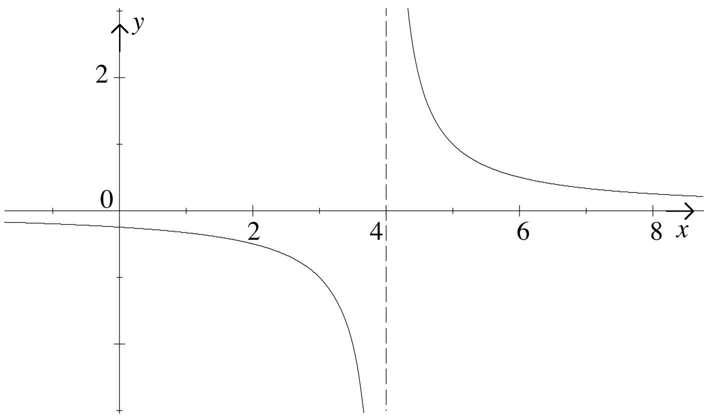
The graph of $y = \sqrt{x-1}$. The domain is all real $x \geq 1$ and the range is all real $y \geq 0$.

b.



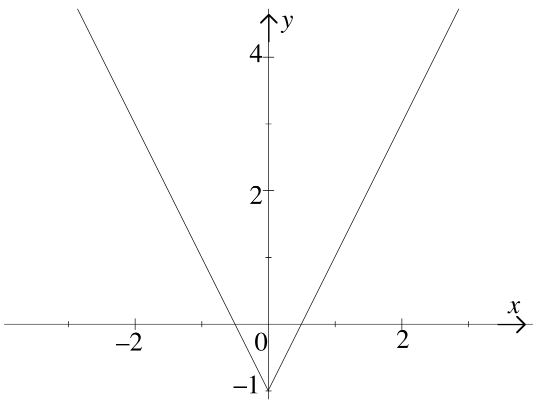
The graph of $y = |2x|$. Its domain is all real x and range all real $y \geq 0$.

c.



The graph of $y = \frac{1}{x-4}$. The domain is all real $x \neq 4$ and the range is all real $y \neq 0$.

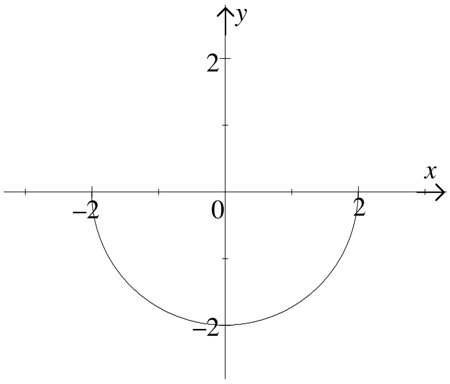
d.



The graph of $y = |2x| - 1$. The domain is all real x, and the range is all real $y \geq -1$.

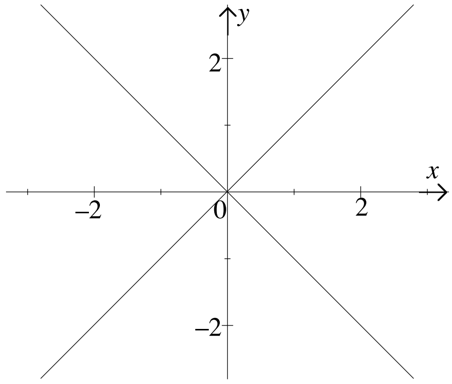
3. $y^2 = x^3$ is not a function. If $x = 1$, then $y^2 = 1$ and $y = 1$ or $y = -1$.

4. **a.**



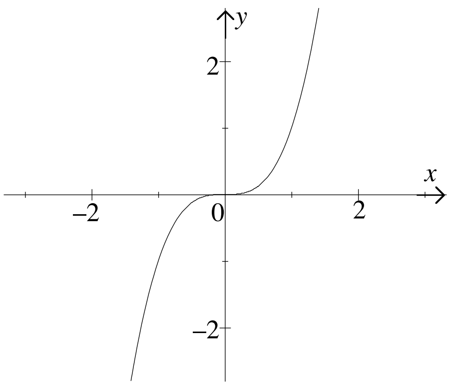
The graph of $y = -\sqrt{4 - x^2}$. This is a function with the domain: all real x such that $-2 \leq x \leq 2$ and range: all real y such that $-2 \leq y \leq 0$.

b.



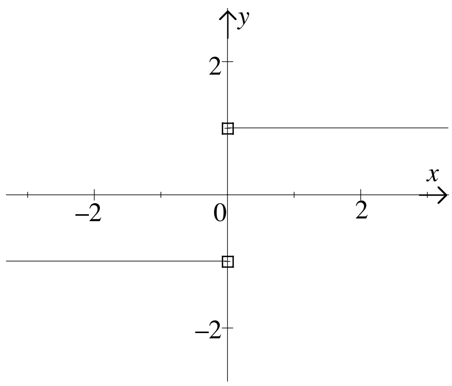
The graph of $|x| - |y| = 0$. This is not the graph of a function.

c.



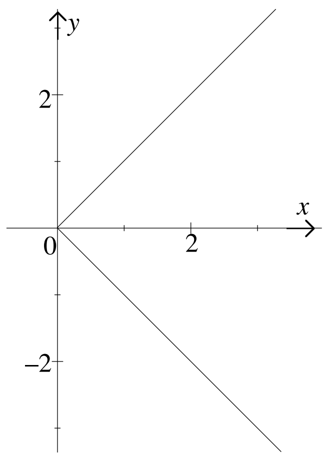
The graph of $y = x^3$. This is a function with the domain: all real x and range: all real y.

d.



The graph of $y = \frac{x}{|x|}$. This is the graph of a function which is not defined at $x = 0$. Its domain is all real $x \neq 0$, and range is $y = \pm 1$.

e.



The graph of $|y| = x$. This is not the graph of a function.

5. **a.** The values of x in the interval $0 < x < 4$ are not in the domain of the function.

b. $x = 1$ and $x = -1$ are not in the domain of the function.