Mathematics Learning Centre

The University of Sydney

An Introduction to Calculus

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Some rules of differentiation and how to use them

Derivatives can be used in economics whenever we want to talk about a rate of change or want to maximise or minimise something. For example, marginal cost is the rate of change of total cost with respect to output, so marginal cost can expressed as the derivative of total cost.

In the following notes, we will give a few rules so that you can find derivatives of simple polynomial functions and then discuss how they can be used.

Notation We will use y to represent our dependent variable and q as our independent variable, so we will differentiate with respect to q.

We may use functional notation and write $y = f(q)$ which means that y is a function of q. So, $y = f(q) + g(q)$ means that y is the sum of two functions of q, one called $f(q)$ and one called $g(q)$.

We will express the derivative of y with respect to q as $\dfrac{dy}{dq}$.

If we use functional notation and write $y = f(q)$ then $\dfrac{dy}{dq} = \dfrac{df}{dq}$. That is, the derivative of y with respect to q can be written as the derivative of f with respect to q.

Some rules of differentiation

Rule 1. If $y = c$, where c is an arbitary constant, then

$$\frac{dy}{dq} = 0.$$

For example, if $p = 2000$, then $\dfrac{dp}{dq} = 0$.

Rule 2. If $y = kq^n$ where n is any number and k is a constant, then

$$\frac{dy}{dq} = k(nq^{n-1}).$$

For example, if $TC = 2q^2$, then $\dfrac{dTC}{dq} = 2(2q^{2-1}) = 4q$.

Rule 3. If $y = f(q) + g(q)$, then

$$\frac{dy}{dq} = \frac{df}{dq} + \frac{dg}{dq}.$$

For example, if $TC = 12q + 2q^2$, then $\dfrac{dTC}{dq} = 12(1q^{1-1}) + 2(2q^1) = 12 + 4q$.

These are not the only rules for differentiation, but if we consider only simple polynomials we can make do with these.

An example of using these rules

Adapted from S.Cheung, Mathematics Review Problems, 2004

A firm's total cost of production (TC) is given as a function of output q by the equation

$$TC = q^3 - 20q^2 + 220q.$$

Find the derivative of TC, marginal cost (MC), as a function of q and evaluate it for $q = 10$.

Solution

$$\begin{aligned} MC &= \frac{dTC}{dq} \\ &= 3q^2 - 20(2q) + 220 \\ &= 3q^2 - 40q + 220. \end{aligned}$$

When $q = 10$, $MC = 3(10)^2 - 40(10) + 220 = 120$.

What does the derivative mean and how can we use it

The derivative of the function $y = f(q)$ gives us a way of calculating the slope of the tangent to the curve at each point q.

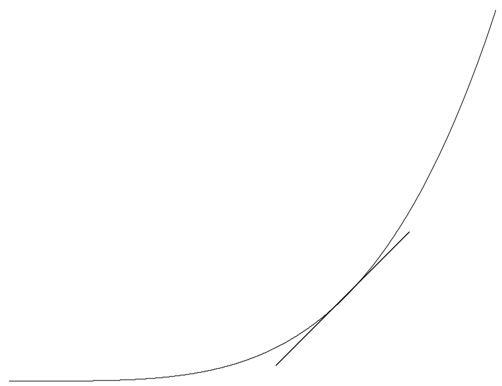

Figure 1.

Figure 1 shows the graph of a function and the tangent to the curve at an arbitrary point. The slope of the tangent line illustrated is given by the derivative of the function evaluated at the arbitrary point.

Therefore, we can use the derivative of a function to determine whether or not it has a maximum or minimum value in the following way.

Figures 2 and 3 show functions with a maximum and a minimum respectively. Note, that at the maximum and minimum, the tangents to the curves are horizontal. That is, the derivatives are zero.

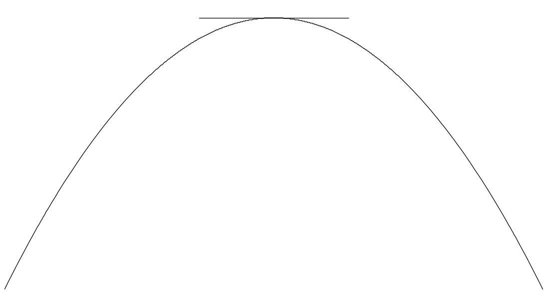

Figure 2.

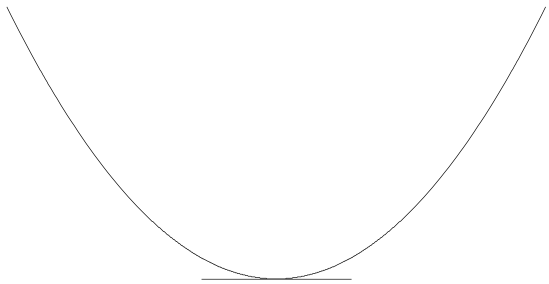

Figure 3.

For simple polynomials, any maximum or minimum will occur when the derivative is zero. A point where the derivative is zero is called a **critical point**.

We now have a way of finding maxima and minima, the first step of which is to find any critical points.

Finding the critical point: an example

Adapted from S.Cheung, Mathematics Review Problems, 2004

The average cost (AC) per unit of output q is given by the equation

$$AC = q^2 - 20q + 220.$$

Find the level of output q at which average cost is a minimum. Find the minimum value of average cost.

Solution

Our average cost function is a simple polynomial and so a minimum will occur only at a critical point.

Therefore, we need the derivative of average cost with respect to q.

$$\frac{dAC}{dq} = 2q - 20.$$

A critical point is a point where the derivative is zero. So,

$$\frac{dAC}{dq} = 0 \quad \text{when} \quad q = 10.$$

We can now conclude that there is only one possible value of q that minimises average cost and it is when $q = 10$.

When $q = 10$, $AC = (10)^2 - 20(10) + 220 = 120$. This is the minimum value for average cost.

Determining the nature of critical points (optional)

In the previous example, there was only one possible value of q that could give us a minimum. In other situations, there could be more than one critical point and so we need a method to decide whether or not our critical point is a maximum or a minimum or neither.

First, let's look at the relationship between the function and the derivative.

In Figure 1, the function is increasing and the slope of the tangent to the curve at the point illustrated is positive. Clearly, if we pick another point on this curve, we will get a tangent with a different slope but the slope will still be positive.

So, if the function $y = f(q)$ is increasing, then the derivative is positive.

Similarly, if we have a function $y = f(q)$ which is decreasing, the derivative is negative. This is illustrated in Figure 4.

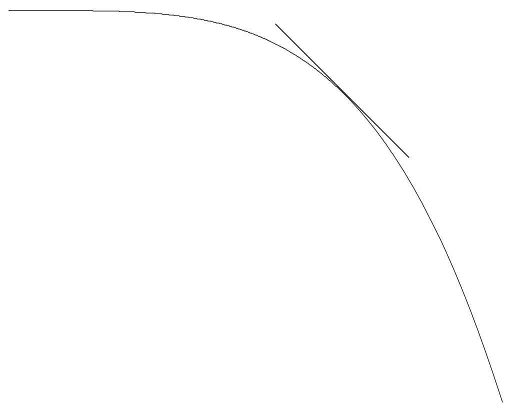

Figure 4.

Figures 1 and 4 indicate to us a way of deducing something about the function if we know about its derivative.

If the derivative is positive for some interval of q values, then the function is increasing for that interval of q values.

And

If the derivative is negative for some interval of q values, then the function is decreasing for that interval of q values.

We can use this to deduce the nature of a critical point.

If the function has a critical point at $q = a$, and for $q < a$ the derivative is positive and for $q > a$ the derivative is negative, then the function has a (local) maximum at $q = a$. This is because for $q < a$, the function is increasing and for $q > a$ the function is decreasing. So, the point $q = a$ must be a maximum. This is illustrated in Figure 5.

Similarly, if the function has a critical point at $q = a$, and for $q < a$ the derivative is negative and for $q > a$ the derivative is positive, then the function has a (local) minimum at $q = a$. This is because for $q < a$, the function is decreasing and for $q > a$ the function is increasing. So, the point $q = a$ must be a minimum. This is illustrated in Figure 6.

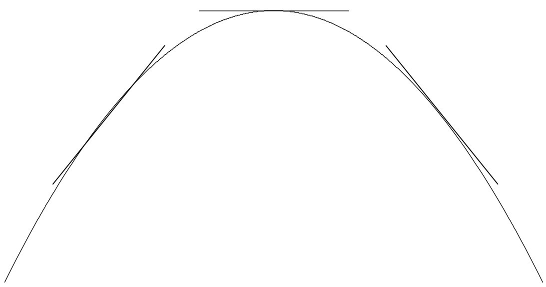

Figure 5.

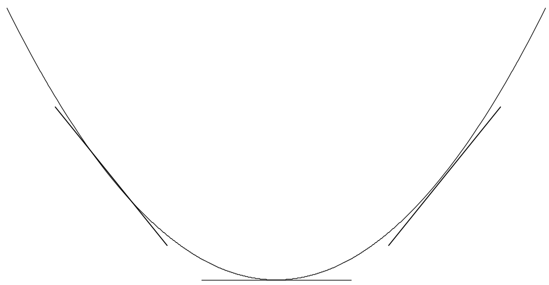

Figure 6.

Another example

Suppose that a firm's profits (which we denote by π) is a function of its output q. Suppose the relationship between these variables is given by

$$\pi = 80q - 4q^2 - 200.$$

For what value of q is the firm's profits maximised. Evaluate π for this value of q.

Solution To maximise profits we differentiate π with respect to q.

$$\frac{d\pi}{dq} = 80 - 8q.$$

The critical point occurs when $\dfrac{d\pi}{dq} = 0$, ie when $80 - 8q = 0$ ie when $q = 10$.

When $q < 10$ (say $q = 9$), $\dfrac{d\pi}{dq} > 0$ so π is increasing for $q < 10$.

When $q > 10$ (say $q = 11$), $\dfrac{d\pi}{dq} < 0$ so π is decreasing for $q > 10$.

Therfore, there is a maximum at $q = 10$. Profit is maximised when $q = 10$, ie the maximum profit is

$$\pi = 80(10) - 4(10)^2 - 200 = 200.$$